Single-Peaked Consistency for Weak Orders Is Easy

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In economics and social choice single-peakedness is one of the most important and commonly studied models for preferences. It is well known that single-peaked consistency for total orders is in P. However in practice a preference profile is not always comprised of total orders. Often voters have indifference between some of the candidates. In a weak preference order indifference must be transitive. We show that single-peaked consistency for weak orders is in P for three different variants of single-peakedness for weak orders. Specifically, we consider Black's original definition of single-peakedness for weak orders, Black's definition of single-plateaued preferences, and the existential model recently introduced by Lackner. We accomplish our results by transforming each of these single-peaked consistency problems to the problem of determining if a 0-1 matrix has the consecutive ones property.

1 Introduction

Single-peakedness is one of the most important and commonly examined domain restrictions on preferences in economics and social choice. The study of single-peaked preferences in computational social choice is often restricted to total orders, but in practical settings voters often have some degree of indifference in their preferences. This is seen in the online repository PrefLib, which contains several datasets comprised of voters with various degrees of partial preferences, many of which are weak orders [26]. Additionally, some election systems are defined for weak orders, e.g., the Kemeny rule [22] and the Schulze rule [30], or can be easily extended for weak orders.

Single-peaked preferences were introduced by Black [5] and they model the preferences of a collection of voters with respect to a one-dimensional axis, i.e., a total ordering of the candidates. Each voter in a single-peaked electorate has a single most preferred candidate (peak) on the axis and the farther that a candidate is from the voter's peak the less preferred they are by the voter. Black extended his model to single-plateaued preferences, which models the preferences of a collection of voters in a similar way, but allows voters to have multiple most preferred candidates (an indifference plateau) in their preferences [6, Chapter 5]. We mention that the definition of single-peaked preferences from Fishburn [18, Chapter 9] for weak orders is the same as Black's definition of single-plateaued preferences.

Elections where the voters have single-peaked preferences over the candidates have many desirable properties in economics and social choice, e.g., the majority relation is transitive [5] and there exist strategy-proof voting rules [28]. Additionally, computational problems often become easier when preferences are single-peaked. For example, when voters in an election have single-peaked (or even *nearly* single-peaked) preferences the complexity of determining if a manipulative action exists often becomes easy [17, 16] and determining the winner for Dodgson and Kemeny elections becomes easy [8] when it is Θ_2^p-complete in general [20, 21].

The problem of single-peaked consistency is to determine if an axis exists such that the preferences of a collection of voters are single-peaked. The first paper to computationally study single-peaked con-

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sistency for partial preferences was Lackner [24], where a partial order is said to be single-peaked with respect to an axis if it can be extended to a total order that is single-peaked with respect to that axis. For clarity we refer to this as existentially single-peaked, or ∃-single-peaked, throughout this paper. Lackner presents algorithms and complexity results for determining the ∃-single-peaked consistency for preference profiles of varying degrees of partial preferences, including top orders, weak orders, local weak orders, and partial orders. Lackner shows that if a given preference profile contains an implicitly specified total order (which is not guaranteed to exist) then ∃-single-peaked consistency for weak orders is in P [24]. Lackner also shows that the general case of ∃-single-peaked consistency for top orders (weak orders with all indifference between last-ranked candidates) is in P [24]. The complexity of the general case of ∃-single-peaked consistency for weak orders was explicitly left as the main open problem in Lackner [24] and we show in this paper that it is in P.

We show that an algorithm to determine if a 0-1 matrix has the consecutive ones property can be used to determine the single-peaked, single-plateaued, and ∃-single-peaked consistency for weak orders *without* requiring an implicitly specified total order. So given a preference profile of weak orders not only can we determine if it is single-peaked, single-plateaued, or ∃-single-peaked, we can find all consistent axes by using the PQ-tree algorithm for determining if a 0-1 matrix has the consecutive ones property [7]. This algorithm was previously used to determine the single-peaked consistency for total orders by Bartholdi and Trick [4] and to determine the single-crossing consistency for total orders by Bredereck et al. [9]. The model of single-crossing preferences is another domain restriction [27] and its corresponding consistency problem for total orders was first shown to be in P by Elkind et al. [12]. We also mention that after single-peaked consistency for total orders was shown to be in P, both Escoffier et al. [15] and Doignon and Falmagne [10] independently found faster direct algorithms.

This paper is organized as follows. In Section 2 we define the types of partial preferences studied, the different variants of single-peakedness, and the consecutive ones matrix problem. We present our results in Section 3, which is split into three sections with each corresponding to a different variant of single-peakedness. Section 3.1 contains our results for ∃-single-peaked preferences, Section 3.2 for single-plateaued preferences, and Section 3.3 for single-peaked preferences. In each of these sections we redefine the variant of single-peakedness using forbidden substructures and describe the transformation from its consistency problem to the problem of determining if a 0-1 matrix has the consecutive ones property. We conclude in Section 4 by summarizing our results and stating some possible directions for future work.

2 Preliminaries

A *preference order*, v, is an ordering of the elements of a finite candidate set, C. A multiset of preference orders, V, is called a *preference profile*. (We sometimes refer to each v as a voter with a corresponding preference order.) A *partial order* is a transitive and reflexive binary relation on a set. A *weak order* is a partial order that is complete. a *top order* is a weak order where all indifference is between candidates ranked last, and a *total order* is a weak order with no indifference between candidates.

Example 1 Given the set of candidates $\{a,b,c,d\}$, an example of a total order is $(a > b > d > c)$, an example of a weak order is $(a \sim c > d > b)$, and an example of a top order is $(a > c > b \sim d)$, where "$\sim$" is used to denote indifference between candidates.

We focus on weak orders since they model natural cases where voters are not able to discern between two candidates or where they view them as truly equal. Allowing each voter to state a weak preference order still requires that they specify each candidate in their order, but gives them the ability to have multiple candidates at each position.

It is very natural for election systems to be defined for weak orders. The Kemeny rule and Schulze rule are defined for weak orders [22, 30], and clearly election systems based on pairwise comparisons (e.g., Copeland) can be used to evaluate a preference profile of partial votes. The Borda count can be extended for top orders [13] and a recent paper has even explored the complexity of the manipulation problem for such extensions to the Borda count and defined additional extensions for election systems to be defined for top orders [29].

2.1 Variants of Single-Peakedness

In our definitions of each variant of single-peakedness we refer to a total ordering of the set of candidates that each preference profile is consistent with as an axis A. Like Bartholdi and Trick [4], who were the first to show single-peaked consistency for total orders in P, we say that a preference order v is strictly increasing (decreasing) along a segment X of A if each candidate in X is strictly preferred to each candidate to its left (right) in X. Similarly, we say that a preference order is increasing (decreasing) along a segment X of A if each candidate in X is strictly preferred or ranked indifferent to each candidate to its left (right) in X. When we say that a preference order v is remaining constant along a segment, then all candidates in that segment are ranked indifferent to each other.

We begin our discussion of single-peaked preferences by stating the definition of single-peaked preferences for total orders. We use the definition found in the work by Bartholdi and Trick [4].

Definition 2 *A preference profile V of total orders is* single-peaked *with respect to an axis A if for every $v \in V$, A can be split at the most preferred candidate (peak) of v into two segments X and Y (one of which can be empty) such that v has strictly increasing preferences along X and v has strictly decreasing preferences along Y.*

We now define each of the three variants of single-peaked preferences for weak orders that we study in this paper and present an example of each in Figure 1.

2.1.1 Single-Peaked Preferences

Single-peaked preferences for weak orders can be defined in the same way as single-peaked preferences for total orders.

Definition 3 *A preference profile V of weak orders is* single-peaked *with respect to an axis A if for every $v \in V$, A can be split at the most preferred candidate (peak) of v into two segments X and Y (one of which can be empty) such that v has strictly increasing preferences along X and v has strictly decreasing preferences along Y.*

Notice that for a weak preference order to be single-peaked it must have a single most preferred candidate and can only contain indifference between at most two candidates at each position. Otherwise the segments X and Y referred to in Definition 3 would not be *strictly* increasing/decreasing. We define the corresponding problem of single-peaked consistency for weak orders below.

Given: A preference profile V of weak orders and a set of candidates C.

Question: Does there exist an axis A such that V is single-peaked with respect to A?

2.1.2 Single-Plateaued Preferences

A slightly weaker restriction than single-peakedness for weak orders is single-plateauedness [6, Chapter 5]. Single-peaked and single-plateaued preferences are closely related domain restrictions and Barberà [2] discusses how the amounts of indifference permitted in these restrictions impact their properties.

Building upon the definition for single-peaked preferences, we state a definition for single-plateaued preferences.

Definition 4 *A preference profile V of weak orders is* single-plateaued *with respect to an axis A if for every $v \in V$, A can be split into three segments X, Y, and Z (X and Z can each be empty) where v's most preferred candidates are Y, v has strictly increasing preferences along X, and v has strictly decreasing preferences along Z.*

We define the corresponding problem of single-plateaued consistency for weak orders below.

Given: A preference profile V of weak orders and a set of candidates C.

Question: Does there exist an axis A such that V is single-plateaued with respect to A?

2.1.3 Existentially Single-Peaked Preferences

So far we have considered the given preference orders as the true preferences of the voters. One approach to dealing with partial preferences is to assume that voters have an underlying total preference order and consider extensions of their preferences to total orders (see, e.g., [23]). This is the approach taken by Lackner for the existential model of single-peakedness [24].

Definition 5 *[24] A preference profile V of weak orders is* $\exists$-single-peaked *with respect to an axis A if for every $v \in V$, v can be extended to a total order v' such that the profile V' of total orders is single-peaked with respect to A.*

We can restate Definition 5 without referring to extensions to better see how it relates to single-peaked and single-plateaued preferences.

Observation 6 *A preference profile V of weak orders is* $\exists$-single-peaked *with respect to an axis A if and only if for every $v \in V$, A can be split into three segments X, Y, and Z (X and Z can each be empty) where v's most preferred candidates are Y, v has increasing preferences along X, and v has decreasing preferences along Z.*

We define the corresponding problem of $\exists$-single-peaked consistency for weak orders below.

Given: A preference profile V of weak orders and a set of candidates C.

Question: Does there exist an axis A such that V is $\exists$-single-peaked with respect to A?

Figure 1 illustrates an example of each variant of single-peakedness for weak orders described where each preference order is consistent with respect to the axis $A = a < d < b < e < c$. In Figure 1 the preference order $(b > d \sim e > c > a)$ is single-peaked, single-plateaued, and $\exists$-single-peaked. The preference order $(a \sim d > b > e > c)$ is single-plateaued and $\exists$-single-peaked, but not single-peaked since it has more than one most preferred candidate. The preference order $(c > b \sim e > d \sim a)$ is $\exists$-single-peaked and not single-plateaued or single-peaked since it is not strictly increasing to its most preferred candidate(s).

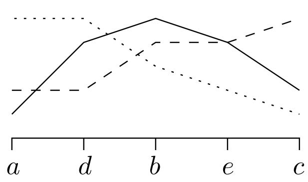

Figure 1: The solid line represents the single-peaked preference order $(b > d \sim e > c > a)$, the dotted line represents the single-plateaued preference order $(a \sim d > b > e > c)$, and the dashed line represents the $\exists$-single-peaked preference order $(c > b \sim e > d \sim a)$.

We conclude our discussion of these variants of single-peakedness for weak orders by stating several observations.

First we show that there exists an $\exists$-single-peaked consistent preference profile that does not have a transitive majority relation. We say that a majority relation is transitive if when $x > y$ and $y > z$ by majority, then $x > z$ by majority. Note that single-peaked and single-plateaued preferences both have transitive majority relations [5, 6].

Consider the preference profile V comprised of the following five voters from Table 9.1 in Fishburn [18].

$$\begin{array}{rl} v_1 & (b > a > c) \\ v_2, v_3 & (c > b > a) \\ v_4, v_5 & (a > b \sim c) \end{array}$$

When we evaluate this preference profile under the simple majority rule where $x > y$ by simple majority if more voters state $x > y$ than $y > x$, then V has the majority cycle $a > c > b > a$ [18]. Clearly V is $\exists$-single-peaked consistent with respect to the axis $A = a < b < c$, so we can make the following observation.

Observation 7 *There exists a preference profile of weak orders that is $\exists$-single-peaked and does not have a transitive majority relation.*

The existential model for single-peakedness considers the existence of a single extension of the preferences of all of the voters to total orders. We briefly consider the case where all extensions to total orders must be single-peaked and make two observations.

Observation 8 *If a preference profile of weak orders is single-peaked then all extensions of the preferences to total orders are also single-peaked.*

Observation 9 *If a preference profile of weak orders is single-plateaued and each preference order has at most two most preferred candidates, then all extensions of the preferences to total orders are single-peaked.*

2.2 Consecutive-Ones Matrices

All of our polynomial-time results are due to transformations to the following problem of determining if a 0-1 matrix has the consecutive ones property.

Given: A 0-1 matrix M.

Question: Does there exist a permutation of the columns of M such that in each row all of the 1's are consecutive?

The above problem was shown to be in P by Fulkerson and Gross [19]. Booth and Lueker [7] improved on this result by finding a linear-time algorithm through the development and use of the novel PQ-tree data structure, which contains all possible permutations of the columns of a matrix such that all of the 1's are consecutive in each row.

3 Results

The following three sections consist of our results and they are structured as follows. We examine each variant of single-peakedness starting with the weakest restriction and ending with the strongest. When we examine each restriction we present an alternative definition of the variant of single-peakedness using forbidden substructures and the transformation to the problem of determining if a 0-1 matrix has the consecutive ones property.

3.1 Existentially Single-Peaked Consistency

The most general of the three variants mentioned in Section 2.1 is the model of ∃-single-peaked preferences. The construction and corresponding proof will be the basis for showing that single-peaked and single-plateaued consistency for weak orders are each also in P.

Given an axis A and a preference order v, if v is ∃-single-peaked with respect to A then v cannot have strictly decreasing and then strictly increasing preferences with respect to A. Following the terminology used by Lackner [24], we refer to this as a v-valley.

Definition 10 *A preference order v over a candidate set C contains a* v-valley *with respect to an axis A if there exist candidates $a,b,c \in C$ such that $a < b < c$ in A and $(a > b)$ and $(c > b)$ in v.*

Using the v-valley substructure we can state the following lemma, which will simplify our argument used in the proof of Theorem 14.

Lemma 11 *[24] Let V be a preference profile of weak orders. V is ∃-single-peaked with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley with respect to A.*

To construct a matrix from a preference profile of weak orders, we apply essentially the same transformation as used Bartholdi and Trick [4] for total orders (see Example 13). We describe the construction below.

Construction 12 *Let V be a preference profile of weak orders over candidate set C. For each $v \in V$ construct a $(\|C\|-1) \times \|C\|$ matrix X_v. Each column of X_v corresponds to a candidate in C. For each candidate $c \in C$ let k be the number of candidates that are strictly preferred to c in v and let the corresponding column in matrix X_v contain k 0's starting at row one, with the remaining entries filled with 1's. All $\|V\|$ of the matrices are row-wise concatenated to yield the $(\|V\| \cdot (\|C\|-1)) \times \|C\|$ matrix X.*

The main difference in our construction is that we have one fewer row in each of the individual preference matrices. In the construction used by Bartholdi and Trick [4], given a preference order v over a set of candidates C, for all $a,b \in C$, $(a > b)$ in v if and only if the number of 1's in the column corresponding to a is greater than the number of 1's in the column corresponding to b in v's corresponding individual preference matrix. Notice that this still holds for our construction.

The polynomial-time results for ∃-single-peaked consistency for weak orders and local weak orders proved in Lackner [24] require that the given preference profile contains a *guiding order*, i.e., an implicitly specified total order. Given a preference profile V, a guiding order can be constructed iteratively in the following way. If there exists a $v \in V$ such that the last ranked candidate in v is not ranked indifferently with any other candidate, then that candidate is appended to the top of the guiding order. This is then repeated on the preference profile restricted to the candidates not yet added to the guiding order until either the guiding order is a total order or there is no $v \in V$ with a unique last ranked candidate, the case where no guiding order exists [24]. Observe that if a given preference profile is ∃-single-peaked then it remains ∃-single-peaked if a guiding order is added as an additional preference order [24]. It is important to point out that our results do not depend on the existence of a guiding order in a preference profile. Below we show how Construction 12 is applied to a preference profile of weak orders that is ∃-single-peaked.

Example 13 Consider the preference profile V that consists of the preference orders v and w. Let the preference order v be $(a \sim c > b > e \sim d > f)$ and the preference order w be $(a > b > c > e \sim d > f)$. Notice that V does not contain a guiding order, which is required by the polynomial-time algorithm for weak orders found in Lackner [24].

$$X_v = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad X_w = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

We then row-wise concatenate X_v and X_w to construct X.

$$X = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad X' = \begin{array}{c} \begin{matrix} b & a & c & d & e & f \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Next, we permute the columns of X so that in each row all of the 1's are consecutive to yield X'. Observe that V is ∃-single-peaked with respect to $b < a < c < d < e < f$, the ordering of the columns of X' as its axis.

We now show that ∃-single-peaked consistency for weak orders and the problem of determining if the constructed 0-1 matrix has the consecutive ones property are equivalent using Lemma 11 and Construction 12.

Theorem 14 *A preference profile V of weak orders is ∃-single-peaked consistent if and only if the matrix X, constructed using Construction 12, has the consecutive ones property.*

Proof. Let V be a preference profile of weak orders. Essentially the same argument as used by Bartholdi and Trick [4] holds.

If V is $\exists$-single-peaked with respect to an axis A then by Lemma 11 we know that no preference order $v \in V$ contains a v-valley with respect to A. When the columns of the matrix X are permuted to correspond to the axis A no row will contain the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$ since this corresponds to a preference order that strictly decreases and then strictly increases along the axis A (a v-valley). Therefore X has the consecutive ones property.

For the other direction suppose that V is not $\exists$-single-peaked, then by Lemma 11 we know that for every possible axis there exists a preference order $v \in V$ such that v contains a v-valley with respect to that axis. So every permutation of the columns of X will correspond to an axis where some preference order has a v-valley. As stated in the other direction, a v-valley corresponds to a row containing the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$ so clearly X does not have the consecutive ones property.

The only difference from the argument used by Bartholdi and Trick [4] for total orders is that in our case the preference orders can remain constant at the peak and at points on either side of the peak. The same argument still applies since by Lemma 11 the absence of v-valleys with respect to an axis is equivalent to a profile of weak orders being $\exists$-single-peaked with respect to that axis. ❑

Corollary 15 *$\exists$-Single-peaked consistency for weak orders is in* P.

3.2 Single-Plateaued Consistency

Single-plateaued preferences are a much more restrictive model than $\exists$-single-peaked preferences since they are essentially single-peaked except that each preference order can have multiple most preferred candidates [6, Chapter 5].

Since a preference order must be strictly increasing and then strictly decreasing with respect to an axis (excluding its most preferred candidates) we can again use the v-valley substructure. However we will need another substructure to prevent two candidates that are ranked indifferent in a voter's preference order from appearing on the same side of that voter's peak (plateau).

Definition 16 *A preference order v over a candidate set C contains a* nonpeak plateau with respect to A *if there exist candidates $a, b, c, \in C$ such that $a < b < c$ in A and either $(a > b \sim c)$ or $(c > b \sim a)$ in v.*

We use the v-valley and nonpeak plateau substructures to state the following lemma.

Lemma 17 *Let V be a preference profile of weak orders. V is single-plateaued with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley with respect to A and no preference order $v \in V$ contains a nonpeak plateau with respect to A.*

Proof. Let C be a candidate set, V be a preference profile of weak orders, and A be an axis.

If V is single-plateaued with respect to A then for every preference order $v \in V$, A can be split into segments X, Y, and Z such that v is strictly increasing along X, remaining constant along Y, and strictly decreasing along Z. Since v is only ever strictly decreasing along Z and Z is the rightmost segment of A, v cannot contain a v-valley with respect to A. For a nonpeak plateau to exist with respect to A there must exist candidates $a, b, c \in C$ such that $a < b < c$ in A and either $(a > b \sim c)$ or $(c > b \sim a)$ in v.

We first consider the case of $(a > b \sim c)$ in v. Since a is strictly preferred to b and c in v and both b and c are to the right of a on the axis we know that both b and c must be in segment Z. However, v is strictly decreasing along Z, so v cannot have a nonpeak plateau of this form.

We now consider the case of $(c > b \sim a)$ in v. Since c is strictly preferred to a and b in v and both a and b are to the left of c on the axis we know that both a and b must be in segment X. However, v is strictly increasing along X, so v cannot have a nonpeak plateau of this form.

For the other direction we consider the case when no preference order $v \in V$ contains a v-valley with respect to A and no preference order $v \in V$ contains a nonpeak plateau with respect to A.

Since no preference order $v \in V$ contains a v-valley with respect to A, we know from Lemma 11 that V is $\exists$-single-peaked with respect to A. Since we also know that no preference order $v \in V$ contains a nonpeak plateau with respect to A it is easy to see that V is single-plateaued with respect to A. ❑

Since the nonpeak plateau substructure is needed in addition to the v-valley substructure, we need to extend Construction 12 so that if a preference order contains a nonpeak plateau with respect to an axis A, then when the columns of its corresponding preference matrix are permuted according to A the matrix will contain a row with the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$.

Notice that if a preference order ranks three candidates indifferent to each other below its peak (plateau) that it will have a nonpeak plateau with respect to *every* possible axis. To handle this case in the extension to Construction 12 we need to ensure that its corresponding preference matrix will contain a row with the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$ for every permutation of its columns.

Construction 18 *Let V be a preference profile of weak orders over candidate set C. For each $v \in V$ construct a $(\|C\| - 1) \times \|C\|$ matrix X_v. Each column of X_v corresponds to a candidate in C. For each candidate $c \in C$ let k be the number of candidates that are strictly preferred to c in v and let the corresponding column in matrix X_v contain k 0's starting at row one, with the remaining entries filled with 1's (as in Construction 12). The following extensions to Construction 12 ensure that if v has nonpeak plateau with respect to an axis A then when the columns of X_v are permuted according to A it will not have consecutive ones in rows.*

For each pair of candidates $a, b \in C$ such that $(a \sim b)$ in v, they are not the most preferred candidates in v, and there is no candidate $c \in C - \{a, b\}$ such that v is indifferent among a, b, and c, then append three additional rows to the matrix X_v where the column corresponding to a is $\begin{bmatrix}0 & 1 & 1\end{bmatrix}'$, the column corresponding to b is $\begin{bmatrix}1 & 1 & 0\end{bmatrix}'$, each column corresponding to a candidate strictly preferred to a and b is $\begin{bmatrix}1 & 1 & 1\end{bmatrix}'$, and each column corresponding to a remaining candidate is $\begin{bmatrix}0 & 0 & 0\end{bmatrix}'$.

If there exist three candidates $a, b, c \in C$ such that $(a \sim b \sim c)$ in v and they are not the most preferred candidates in v, then output a matrix that has no solution.

After constructing an X_v matrix for each $v \in V$, all $\|V\|$ of the matrices are row-wise concatenated to yield a matrix X, except in the case where the input resulted in a matrix with no solution.

We now show how Construction 18 is applied to a preference profile of weak orders that is single-plateaued.

Example 19 We consider the same preference profile as in Example 13 and we bold the additional rows in this example. Let the preference order v be $(a \sim c > b > e \sim d > f)$ and the preference order w be $(a > b > c > e \sim d > f)$.

$$
\begin{array}{cc}
 & \begin{matrix} a & b & c & d & e & f \end{matrix} \\
X_v = & \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{matrix} a & b & c & d & e & f \end{matrix} \\
X_w = & \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\end{array}
$$

We then row-wise concatenate X_v and X_w to construct X.

$$
X = \begin{array}{c}
\begin{matrix} a & b & c & d & e & f \end{matrix} \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0}
\end{bmatrix}
\end{array}
\qquad
X' = \begin{array}{c}
\begin{matrix} e & b & a & c & d & f \end{matrix} \\
\begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0}
\end{bmatrix}
\end{array}
$$

Next, we permute the columns of X such that in each row all of the ones are consecutive to yield X'. Observe that V is single-plateaued with respect to this new ordering $e < b < a < c < d < f$ as its axis. Also notice that an axis containing d and e adjacent to each other (as seen in Example 13) would not correspond to an ordering of the columns of X with consecutive ones in rows due to the additional rows from the extensions made to Construction 12 in Construction 18.

Construction 12 ensures that no preference order contains a v-valley and the extensions made in Construction 18 ensure that no preference order contains a nonpeak plateau. So the proof of the following theorem uses a similar argument to the proof of Theorem 14. Now the presence of v-valleys *or* nonpeak plateaus, not just v-valleys, is equivalent to a row containing the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$.

Theorem 20 *A preference profile V of weak orders is single-plateaued consistent if and only if the matrix X, constructed using Construction 18, has the consecutive ones property.*

Proof. Let V be a preference profile of weak orders. We extend the argument used by Bartholdi and Trick [4] and the proof of Theorem 14 except in this case we use Lemma 17 instead of Lemma 11.

If V is single-plateaued with respect to an axis A then by Lemma 17 we know that no $v \in V$ contains a v-valley with respect to A and no $v \in V$ contains a nonpeak plateau with respect to A. When the columns of the matrix X are permuted to correspond to the axis A no row will contain the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$ since this would correspond to a preference order that strictly decreases and then strictly increases along the axis A (a v-valley) or it would correspond to a preference order that has two candidates ranked indifferent appearing on the same side of its peak (a nonpeak plateau). Therefore X has the consecutive ones property.

If V is not single-plateaued then we know from Lemma 17 that for every possible axis there exists a preference order $v \in V$ such that v contains a v-valley or v contains a nonpeak plateau with respect to that axis. So every permutation of the columns of X will correspond to an axis where a preference order has a v-valley or a nonpeak plateau. As stated in the other direction, the presence of a v-valley or a nonpeak plateau corresponds to a row containing the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$. Therefore X does not have the consecutive ones property. ❑

Corollary 21 *Single-plateaued consistency for weak orders is in* P.

3.3 Single-Peaked Consistency

We now present our results for the strongest domain restriction on weak orders that we examine, single-peaked preferences. Recall that a preference order is single-peaked with respect to an axis A if it is strictly increasing to a single most preferred candidate (peak) and then strictly decreasing with respect to A. So we again use the v-valley substructure, but like the previous case of single-plateaued preferences we need an additional substructure. Even if no preference order has a v-valley with respect to A it may not be single-peaked because it is indifferent between two candidates on the same side of its peak or has more than one most preferred candidate.

We can handle the first condition just mentioned with the nonpeak plateau substructure used in Section 3.2, but the second condition requires us to view *any* plateau as a forbidden substructure.

Definition 22 *A preference order v over a candidate set C contains a* plateau *with respect to an axis A if there exist candidates $a, b \in C$ such that a and b are adjacent in A and $(a \sim b)$ in v.*

We can now use the plateau substructure and the v-valley substructure to state the following lemma.

Lemma 23 *Let V be a preference profile of weak orders. V is single-peaked with respect to an axis A if and only if no preference order $v \in V$ contains a v-valley with respect to A and no preference order $v \in V$ contains a plateau with respect to A.*

Proof. Let C be a candidate set, V be a preference profile of weak orders, and A be an axis.

If V is single-peaked with respect to A then clearly V is also single-plateaued with respect to A. So by Lemma 17 we know that no preference order $v \in V$ contains a v-valley with respect to A and no preference order $v \in V$ contains a nonpeak plateau with respect to A. Since V is single-peaked we also know that no preference order $v \in V$ has more than one most preferred candidate so clearly no preference order $v \in V$ contains a plateau with respect to A.

For the other direction we consider the case when no preference order $v \in V$ contains a v-valley with respect to A and no preference order $v \in V$ contains a plateau with respect to A.

Since no preference order $v \in V$ contains a v-valley with respect to A we know from Lemma 11 that V is $\exists$-single-peaked with respect to A. Since we also know that no preference order $v \in V$ contains a plateau with respect to A it is easy to see that V is single-peaked with respect to A. ❑

We extend Construction 18 so that if a preference order contains a plateau with respect to an axis A, then when the columns of its preference matrix are permuted according to A the matrix will contain the sequence $\cdots 1 \cdots 0 \cdots 1 \cdots$. Since Construction 18 already ensures this for the case of nonpeak plateaus, our extended construction below only needs to add a condition for plateaus that contain the most preferred candidates in a given preference order.

Construction 24 *Follow Construction 18 except add the following condition while constructing a preference matrix X_v for each preference order $v \in V$.*

If there exist two candidates $a, b \in C$ such that $(a \sim b)$ in v and they are the most preferred candidates in v, then output a matrix that has no solution.

Clearly the extension to Construction 18 above ensures that if there are multiple most preferred candidates in a preference order then the preference matrix constructed from that order does not have the consecutive ones property.

When a preference order has a unique most preferred candidate and is single-plateaued, it is clearly also single-peaked. Construction 24 ensures that no preference order contains more than one most preferred candidate the same way that Construction 18 ensures that no preference order contains three or more candidates that are all ranked indifferent to each other and that are not the most preferred candidates, since this always results in a nonpeak plateau. So the proof of the following theorem follows from the proof of Theorem 20, but using Lemma 23 instead of Lemma 17.

Theorem 25 *A preference profile V of weak orders is single-peaked consistent if and only if the matrix X, constructed using Construction 24 has the consecutive ones property.*

Corollary 26 *Single-peaked consistency for weak orders is in* P.

4 Conclusions and Future Work

We presented three different variants of single-peaked preferences for weak orders and showed that each of their corresponding consistency problems are in P. Since we accomplished this by using transformations to the problem of determining if a 0-1 matrix has the consecutive ones property we are able to apply the PQ-tree algorithm from Booth and Lueker [7]. Using this algorithm we can actually go further than just determining the consistency problem for each of these variants and find *all* consistent axes. An interesting open direction is how the consecutive ones matrix problem relates to other domain restrictions and what benefits there are to having all consistent axes for a given preference profile.

The existential approach introduced by Lackner for single-peaked preferences [24] has been recently applied to other domain restrictions. The model of single-crossing preferences [27] was studied in the existential model by Elkind et al. [11] and the model of top-monotonic preferences [3] was studied in the existential model by Aziz [1]. An interesting direction for future work would be to apply the existential model to other domain restrictions.

Single-peaked preferences are studied because they are a simply stated and important domain restriction that gives insight into how the voters view the candidates and elections with single-peaked voters have nice properties. However, experimental study suggests that in real-world settings voters are often not single-peaked [25], but in this study the single-peaked results only used Black's definition for total orders. It would be interesting to see if real-world datasets of weak orders contain voters that are single-peaked, single-plateaued, or $\exists$-single-peaked.

In single-peaked and nearly single-peaked elections computational problems often become easier [17, 16]. As mentioned by Lackner [24] an important open problem is to determine what computational benefits are gained when a preference profile is $\exists$-single-peaked or even nearly $\exists$-single-peaked. There are several different types of nearly single-peakedness and determining if a given preference profile is nearly single-peaked with respect to a certain distance measure is an interesting computational problem [14]. It would be interesting to see how preference profiles of weak orders impact the complexity of nearly single-peakedness or, as also mentioned by Lackner [24], nearly single-peakedness in the existential model.

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